

MODULI OF LINEAR SECTIONS OF A GENERAL HYPERSURFACE

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0 Introduction and main results. One way to create new projective varieties from a given variety $X \subset \mathbf{P}^r$ is to intersect with linear spaces. It is only natural to want to understand the collection of varieties obtained in this way.

There are different approaches to assigning “moduli” to the linear sections of X . Algebraically, we may assign a point in an appropriate Hilbert scheme H . This works locally as we perturb the linear section, but in order to produce a globally defined variation, we must assign the corresponding point in the quotient of the Hilbert scheme by the action of PGL . The usual technical issues with GIT emerge from this approach, but ultimately one can construct an algebraic moduli map:

$$\mu_m : \mathbf{G}(m, r) \dashrightarrow H / \mathrm{PGL}$$

which, to a general m -plane Λ assigns the variety $[X \cap \Lambda]$. Over the complex numbers, the Hodge structure on the middle cohomology of a linear section $X \cap \Lambda$ provides another way of attaching moduli to the section. Again, under ideal conditions, one gets a period map:

$$\pi_m : \mathbf{G}(m, r) \dashrightarrow \mathcal{D} / \Gamma$$

In either case, whether algebraic or analytic, we call these maps *moduli maps*. Three questions of particular interest are:

- (1) What is the dimension of the image of a moduli map?
- (2) When is a moduli map generically injective?
- (3) If a moduli map is generically finite, what is its degree?

These questions, especially the first two, are certainly not new, and have held the attention of many authors. The first question is infinitesimal in nature – one has to compute the rank of the differential of the moduli map. The second question is global in nature, and the third question is enumerative.

Although the local and global analyses of moduli maps would seem mostly disconnected, the seminal work of Donagi [Don83] shows that the infinitesimal variation of Hodge structure (IVHS) of a variety can, in many situations, recover the variety itself. From the perspective of varying linear slices, we may recast Donagi’s result as proving the generic injectivity of the moduli map for hyperplane sections of the d -uple Veronese varieties $X \subset \mathbf{P}^r$. In the complex analytic setting, such generic injectivity results are called *weak Torelli* or *generic Torelli* theorems.

In this paper, we prove a generic Torelli theorem for the family of m -plane slices of a generic hypersurface X . Numerous generic Torelli statements exist in the literature, including results for complete intersections in [Kon91] and [Ter90], but the problem of generic Torelli for linear sections of a fixed hypersurface seems to have fallen through the cracks.

Question (1) is easy to answer in our setting – a simple first-order calculation left to the reader shows that the images of all moduli maps are as large as possible. We note, however, that this question is easy only because of the nebulous requirement that X be “generic”. Once we try to

qualify what “generic” means, question (1) becomes much more challenging. For example, after considerable effort, in [HMP98] it is shown (among other things) that if we let X be an arbitrary smooth hypersurface of degree d , then, provided r is very very large, the moduli maps for linear slices has image as large as possible.

On the other extreme, an interesting result of Beauville [Bea90] characterizes the degree d smooth hypersurfaces X for which the moduli map of hyperplane sections is constant – this only happens when $d = 2$, or when X is a Fermat hypersurface and $d - 1$ is a power of the characteristic of the (algebraically closed) ground field.

We let $\text{Hyp}(d, m)$ denote the moduli space of degree d hypersurfaces in \mathbf{P}^m , modulo projective equivalence.

Theorem 0.1. *Assume $d > 2r + 1$, and let $X \subset \mathbf{P}^r$ be a general hypersurface of degree d . Then the moduli maps $\mu_m : \mathbf{G}(m, r) \dashrightarrow \text{Hyp}(d, m)$ are generically injective for all m .*

The generic Torelli theorems mentioned earlier are proven using variants of Donagi’s approach. Our technique for proving Theorem 0.1 is completely different; we deduce the result by a simple reduction to the case $m = 1$ of lines.

In the case of lines, we study the moduli map μ_1 by resolving its indeterminacy (at some special points), and then computing its degree at special points in the blown up domain. The points we consider are deep in the locus of indeterminacy, so the details of the resolution require special attention. We are able to carry out the resolution of indeterminacy because of an elementary, but fundamental *versality* result (Corollary 1.28) which essentially allows us to write the moduli map explicitly. This versality result is interesting in its own right, and should be true in a much larger context. Such a generalization will be the subject of future work.

Our analysis yields complete solutions to questions (2) and (3) in the case of lines. In order to answer question (2), we study the monodromy on the set of k -incident lines to X , where $k = d - (2r - 2)$. These are lines which meet X at k points, i.e. *as infrequently as possible*.

The set of k -incident lines to X divides itself into groups according to the multiplicity vector (m_1, \dots, m_k) , $\sum_{i=1}^k m_i = d$, describing the way the lines meet X . Clearly, monodromy preserves the multiplicity vector of a k -incident line.

We prove the following generalization of a monodromy result on flex lines of plane curves found in [Har79]:

Theorem 0.2. *Let X be a general hypersurface of degree $d \geq 2r$, let $k = d - (2r - 2)$, and let (m_1, \dots, m_k) be a multiplicity vector with at least two distinct entries.*

Then the monodromy group of the set of k -incident lines to X with multiplicity vector (m_1, \dots, m_k) is the full symmetric group.

The key to answering question (3) for the moduli map μ_1 is to analyze the *tri-incident* lines of X – these are lines meeting X at exactly three points.

Theorem 0.3. *Let $X \subset \mathbf{P}^r$ be a general hypersurface of degree $d = 2r + 1$. Then*

$$\deg \mu_1 = 2 \sum_{a \geq b > 1} n_{a,b,1} + 4n_{d-2,1,1}$$

where $n_{i,j,k}$ is the number of tri-incident lines to X with intersection multiplicities (i, j, k) .

Remark 0.4. Theorem 0.1, along with Donagi’s theorem on the generic injectivity of the period map $\text{Hyp}(d, m) \dashrightarrow \mathcal{D}/\Gamma$, implies that the composite period map $\pi_m : \mathbf{G}(m, r) \dashrightarrow \mathcal{D}/\Gamma$ is injective for a generic degree d hypersurface X with $d > 2r + 1$.

Remark 0.5. Theorem 0.1 does not exhaust all cases where $\dim \mathbf{G}(m, r) < \dim \text{Hyp}(d, m)$. We also remark that a similar argument as found in the proof of Theorem 0.1 shows that knowledge of generic injectivity for m -planes implies the same for larger planes.

Remark 0.6. Clearly, some sort of genericity condition on X is required for an affirmative answer to question (2): Consider X a Fermat hypersurface.

Interestingly, the example of the Fermat quintic in \mathbf{P}^2 shows that a genericity assumption stronger than “smooth” is required for a uniform answer to question (3). In [CL08], the degree of μ_1 for a general quintic is 420 (in agreement with Theorem 0.3), while for the Fermat quintic the degree drops to 150.

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0.2 Conventions. We work over an algebraically closed field of characteristic zero. If F is a sheaf, we let $\Gamma(F)$ denote the space of global sections of F .

1 Versality.

1.1 Local models. Throughout this section $X \subset \mathbf{P}^r$ will be a smooth hypersurface not containing any lines.

Definition 1.1. We let $\pi : \mathbf{P} \longrightarrow \mathbf{G}(1, r)$ denote the tautological \mathbf{P}^1 -bundle.

Definition 1.2. The *point-line incidence correspondence* for X is the variety $\Sigma \subset \mathbf{P}$ defined by

$$\Sigma := \{(x, [\ell]) \in X \times \mathbf{G}(1, r) \mid x \in \ell\}$$

We let φ, π be the projections of Σ to the first and second factors.

Lemma 1.3. Σ is a smooth variety.

Proof. The projection $\varphi : \Sigma \longrightarrow X$ is a projective bundle, with fibers isomorphic to \mathbf{P}^{r-1} . □

Clearly, since X is assumed to contain no lines, the map $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ is a finite branched cover of degree d .

Definition 1.4. We let W_n , ($n > 1$), denote the pair $(Y_n, \mathbf{P}^1 \times \mathbf{A}^{n-1} \longrightarrow (\mathbf{A}^{n-1}, 0))$, with $Y_n \subset \mathbf{P}^1 \times \mathbf{A}^{n-1}$ defined by the equation

$$s^n + a_{n-2}s^{n-2}t^2 + a_{n-3}s^{n-3}t^3 + \cdots + a_0t^n = 0,$$

where a_i are coordinates on \mathbf{A}^{n-1} , and $[s : t]$ are homogeneous coordinates on \mathbf{P}^1 .

Remark 1.5. To simplify notation, we will often write the affine equation

$$z^n + a_{n-2}z^{n-2} + \cdots a_0 = 0$$

for the equation defining W_n . W_n is often referred to as the “mini-versal unfolding” of the singularity $z^n = 0$.

Proposition 1.6. Let B be a smooth variety, $b \in B$ a marked point. Suppose $Z \subset \mathbf{P}^1 \times B$ is a divisor such that $Z_b \subset \mathbf{P}^1 \times \{b\}$ is a scheme isomorphic to $s^n = 0$. Then there is an étale neighborhood U of b and a map $\mu : (U, b) \longrightarrow (\mathbf{A}^{n-1}, 0)$ such that $Z_U \subset \mathbf{P}^1 \times U$ is isomorphic to the pullback of W_n under μ .

Proof. Since B is smooth, it is étale locally isomorphic to \mathbf{A}^m . So we may reduce to the case $(B, b) = (\mathbf{A}^m, 0)$.

After passing to an open set, the set Z is defined by a single equation

$$s^n + u_{n-1}s^{n-1}t + \cdots + u_0t^n = 0$$

where $u_i \in \mathfrak{m}_0 \subset \Gamma(O_{\mathbf{A}^m})$. (Here \mathfrak{m}_0 is the maximal ideal of the origin.) We may “complete the n -th power” and find an isomorphic $Z' \subset \mathbf{P}^1 \times \mathbf{A}^m$ defined by a similar equation, but with vanishing u_{n-1} term.

The $n-1$ functions $u_i \in \mathfrak{m}_0 \subset \Gamma(O_{\mathbf{A}^m})$, $i = 0, \dots, n-2$ define a map $(\mathbf{A}^m, 0) \longrightarrow (\mathbf{A}^{n-1}, 0)$ with the desired property. \square

Proposition 1.7. *Let (B, b) be as in the previous proposition, but now suppose $Z \subset \mathbf{P}^1 \times B$ is a divisor whose restriction to $Z_b \subset \mathbf{P}^1 \times \{b\}$ is of the form $n_1p_1 + \cdots + n_ip_i$ with each $n_j \geq 1$.*

Then there exists an étale open neighborhood U of b such that Z_U is a disjoint union $\coprod_j Z_j$, and furthermore there exists a map $(U, b) \longrightarrow (\prod_j \mathbf{A}^{n_j-1}, 0)$ such that Z_j is isomorphic to the pullback of W_{n_j} under the composite $(U, b) \longrightarrow (\prod_j \mathbf{A}^{n_j-1}, 0) \longrightarrow (\mathbf{A}^{n_j-1}, 0)$.

Proof. The argument is completely similar to the proof of Proposition 1.6. \square

Remark 1.8. The maps to W_n in these propositions are not unique, however it is easy to show that they induce a unique map on the tangent space of B at b .

Definition 1.9. Let (B, b) be a smooth variety with marked point, and let $Z \subset \mathbf{P}^1 \times B$ be a closed subset such that Z_b is as in Proposition 1.7. Then $Z \subset \mathbf{P}^1 \times B \longrightarrow B$ is *versal at b* if any (and hence all) of the maps $(U, b) \longrightarrow (\prod_j \mathbf{A}^{n_j-1}, 0)$ is a local submersion, i.e. the map on tangent spaces is surjective.

Definition 1.10. Let (B, b) be a smooth variety with marked point, and let $Z \subset \mathbf{P}^1 \times B$ be a closed subset such that Z_b is as in Proposition 1.7. Then $Z \subset \mathbf{P}^1 \times B$ is *mini-versal at b* if any (and hence all) of the maps $(U, b) \longrightarrow (\prod_j \mathbf{A}^{n_j-1}, 0)$ is a local isomorphism.

1.2 Lines meeting X infrequently.

Notation and Setup 1.11. Let $\sum_i^k m_i = d$, $\sum_{i=1}^j n_i = d$, and let \mathbf{P}^N denote the projective space of degree d hypersurfaces in \mathbf{P}^r . We define the following three types of incidence correspondences:

(1)

$$I_k(m_1, \dots, m_k) = \overline{\left\{ (X, \ell) \mid X \cap \ell = \sum_i m_i p_i, \text{ for distinct } p_i \in \ell \right\}} \subset \mathbf{P}^N \times \mathbf{G}(1, r)$$

(2)

$$I'_k(m_1, \dots, m_k) = \overline{\left\{ (X, \ell, p_1, \dots, p_k) \mid X \cap \ell = \sum_i m_i p_i \right\}} \subset \mathbf{P}^N \times \mathbf{P} \times_{\mathbf{G}(1, r)} \cdots \times_{\mathbf{G}(1, r)} \mathbf{P}$$

(3)

$$I''_{k,j}(m_1, \dots, m_k \mid n_1, \dots, n_j) = \overline{\left\{ (X, \ell, p_1, \dots, p_k, \ell', q_1, \dots, q_j) \mid X \cap \ell = \sum_i m_i p_i, X \cap \ell' = \sum_i n_i q_i \right\}}$$

The closures are taken of the sets where the objects parametrized are all distinct (for points) and do not intersect (for lines).

1.2.1 Lemmas on correspondences.

Lemma 1.12. $I_k(m_1, \dots, m_k)$ is irreducible for all k .

Proof. Consider the correspondence $I'_k(m_1, \dots, m_k)$. The projection to the data (ℓ, p_1, \dots, p_k) makes I'_k into a projective space bundle over the parameter space of tuples (ℓ, p_1, \dots, p_k) , and hence is irreducible. $I'_k(m_1, \dots, m_k)$ dominates $I_k(m_1, \dots, m_k)$ by forgetting the data of the marked points (p_1, \dots, p_k) . The lemma follows. \square

Lemma 1.13. The varieties $I''_{k,j}(m_1, \dots, m_k \mid n_1, \dots, n_j)$ are irreducible for all k, j , and all multiplicities $(m_1, \dots, m_k), (n_1, \dots, n_j)$.

Proof. The general fiber of the projection

$$(X, \ell, p_1, \dots, p_k, \ell', q_1, \dots, q_j) \mapsto (\ell, p_1, \dots, p_k, \ell', q_1, \dots, q_j)$$

is a linear space in \mathbf{P}^N with dimension independent of the pair of non-intersecting lines ℓ, ℓ' . This follows because the restriction map

$$\Gamma(\mathcal{O}_{\mathbf{P}^r}(d)) \longrightarrow \Gamma(\mathcal{O}_{\ell \cup \ell'}(d))$$

is surjective. The lemma follows at once. \square

Definition 1.14. Let $X \subset \mathbf{P}^r$ be a hypersurface. A line ℓ is k -incident to X if the set-theoretic intersection $X \cap \ell$ consists of k points.

Proposition 1.15. Assume $k = d - (2r - 2)$ is positive. Then a general hypersurface $X \subset \mathbf{P}^r$ of degree d has finitely many k -incident lines, and has no k' -incident lines for $k' < k$.

Proof. This is standard, and left to the reader. \square

1.2.2 First-order analysis near a k -incident line. Let $(X, \ell) \in I_k(m_1, \dots, m_k)$ be a general point. We will study the first order deformations of $X \cap \ell$, as we vary ℓ .

Fix homogeneous coordinates $[x_0, \dots, x_r]$ for \mathbf{P}^r such that ℓ is given by $x_0 = \dots = x_{r-2} = 0$. Let X be given by the degree d equation

$$F = P(x_{r-1}, x_r) + \sum_{j=0}^{d-2} x_j G_j$$

where

$$P = \prod_{i=1}^k (x_{r-1} - \alpha_i x_r)^{m_i}.$$

Definition 1.16. Let $G \subset \mathrm{PGL}_{r+1}$ be the group of matrices of the form:

$$G = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{r-1,0} & a_{r-1,1} & \dots & 1 & 0 \\ a_{r,0} & a_{r,1} & \dots & 0 & 1 \end{bmatrix}$$

where $a_{i,j}$, $(i = r-1, r), (j = 0, \dots, r-2)$ are any constants.

Remark 1.17. The orbit $G \cdot [\ell]$ is an open subset of $\mathbf{G}(1, r)$ isomorphic to \mathbf{A}^{2r-2} .

Definition 1.18. Let $Z \subset \ell \times G$ be the union

$$\bigcup_{g \in G} ((g \cdot X) \cap \ell, g).$$

Lemma 1.19. The variety $Z \subset \ell \times G$ is isomorphic to the variety $\Sigma \subset \mathbf{P}$ above the open set $G \subset \mathbf{G}(1, r)$.

Proof. This is clear. □

Our goal here is to understand the first-order infinitesimal neighborhood of the family $Z \subset \ell \times G \longrightarrow G$ near the point $[\ell] \in G$.

Definition 1.20. We let g_j be the polynomial in x_{r-1}, x_r which is the restriction of G_j to ℓ .

Lemma 1.21. Let $\{c_{i,j}\}, (i = r-1, r), (j = 0, \dots, r-2)$ be the elements $\partial_{a_{i,j}} \in T_{[\ell]}G$.

The first-order family $Z_\varepsilon \subset \ell \times T_{[\ell]}G \longrightarrow T_{[\ell]}G$ is isomorphic to the zero locus of:

$$P(x_{r-1}, x_r) + \sum_{j=0}^{r-2} (c_{r-1,j}x_{r-1} + c_{r,j}x_r) g_j.$$

Proof. The first-order family parametrized by the Lie algebra of the group G is given by applying the infinitesimal substitutions

$$\begin{aligned} x_0 &\mapsto x_0 + \varepsilon(c_{r-1,0}x_{r-1} + c_{r,0}x_r) \\ x_1 &\mapsto x_1 + \varepsilon(c_{r-1,1}x_{r-1} + c_{r,1}x_r) \\ &\vdots \\ x_{r-1} &\mapsto x_{r-1} \\ x_r &\mapsto x_r \end{aligned}$$

to the equation F defining X and then restricting to ℓ . The constants $\{c_{i,j}\}$ vary freely, and represent coordinates on the Lie algebra of G . The end result is the family given in the statement of the lemma. □

Remark 1.22. It will usually be more convenient to use the affine notation

$$p(z) + \sum_{j=0}^{r-2} (c_{r-1,j}z + c_{r,j}) g_j(z).$$

Using the infinitesimal analysis above, we will now prove some lemmas which will allow us to understand the versality of the point-line correspondence $\Sigma \longrightarrow \mathbf{G}(1, r)$ near a k -incident line ℓ , where $k = d - (2r - 2)$.

1.3 Some lemmas on multiplication of polynomials.

Notation and Setup 1.23. Throughout subsection 1.3, we will keep the following notation.

- (1) Suppose n_1, \dots, n_k are positive integers such that $\sum_{i=1}^k n_i = d$ and $\sum_{i=1}^k (n_i - 1) = 2r - 2$ for some positive r .
- (2) Let P_m denote the vector space of polynomials in one variable z with degree $\leq m$.
- (3) Finally, let $\alpha_1, \dots, \alpha_k \in \mathbf{A}^1 = \text{Spec } k[z]$ be distinct points.

1.3.1 First Lemma.

Lemma 1.24. *There exists an $r - 1$ -dimensional subspace $V_{r-1} \subset P_{d-1}$ such that the multiplication-then-restriction map*

$$\rho : P_1 \otimes V_{r-1} \longrightarrow \bigoplus_{i=1}^k k[z]/(z - \alpha_i)^{n_i-1}$$

is an isomorphism.

Proof. We will explicitly construct a vector space V_{r-1} with the desired property.

Our construction is slightly different depending on the parities of the numbers n_i . So, for the convenience of the reader, let us first assume all n_i are odd.

Then for each i , we define the polynomial

$$p_i := \prod_{j \neq i} (z - \alpha_j)^{n_j-1}.$$

Next, for each i consider the vector space

$$A_i := \text{Span} \{ p_i \cdot 1, p_i \cdot (z - \alpha_i)^2, p_i \cdot (z - \alpha_i)^4, \dots, p_i \cdot (z - \alpha_i)^{n_i-3} \}.$$

Then it is immediately verified that

$$\rho(P_1 \otimes A_i) = (0, \dots, k[z]/(z - \alpha_i)^{n_i-1}, 0, \dots).$$

Note that $\dim A_i = (n_i - 1)/2$. By letting $V_{r-1} = A_1 + \dots + A_k$, the lemma follows in this case.

Now suppose n_1 is even. Since $\sum (n_i - 1)$ is even, we may assume without loss of generality that n_2 is also even. Then we define the polynomials

$$\begin{aligned} q_1 &= (z - \alpha_2)^{n_2-2} \prod_{j \neq 1,2} (z - \alpha_j)^{n_j-1} \\ q_2 &= (z - \alpha_1)^{n_1-2} \prod_{j \neq 1,2} (z - \alpha_j)^{n_j-1}. \end{aligned}$$

Next set

$$B_{1,2} := \text{Span} \{ q_1, q_1(z - \alpha_1)^2, q_1(z - \alpha_1)^4 \dots q_1(z - \alpha_1)^{n_1-2} = q_2(z - \alpha_2)^{n_2-2}, \dots, q_2(z - \alpha_2)^2, q_2 \}.$$

Then it is straightforward to verify that

$$\rho(P_1 \otimes B_{1,2}) = (k[z]/(z - \alpha_1)^{n_1-1}, k[z]/(z - \alpha_2)^{n_2-1}, 0, 0, \dots).$$

Note that $\dim B_{1,2} = (n_1 + n_2 - 2)/2$. By pairing up the remaining even n_i and creating the analogous vector spaces $B_{i,j}$, and then for odd n_i using the spaces A_i , one can check that

$$V_{r-1} := \sum_{n_i \text{ odd}} A_i + B_{1,2} + \dots$$

has the desired property, i.e. that ρ is an isomorphism. \square

1.3.2 Second Lemma.

Lemma 1.25. *Maintain the setup before the previous lemma. Then there exists an $r-1$ -dimensional subspace $V_{r-1} \subset P_{d-1}$ such that the multiplication-then-restriction map*

$$\rho' : P_1 \otimes V_{r-1} \longrightarrow k[z]/(z - \alpha_1)^{n_1} \oplus_{i=2}^k k[z]/(z - \alpha_i)^{n_i-1}$$

has the following properties:

- (1) ρ' is injective.
- (2) The image of ρ' does not contain the element

$$w = ((z - \alpha_1)^{n_1-1}, 0, 0, \dots).$$

Proof. The spaces V_{r-1} constructed in the proof of Lemma 1.24 satisfy these two properties. We leave the verification to the reader. \square

Remark 1.26. The conditions in these two lemmas are open – a general $V_{r-1} \subset P_{d-1}$ will have the desired properties in each lemma.

Theorem 1.27. *Let $\sum_{i=1}^k m_k = d$, and suppose $\sum_i m_i - 1 = 2r - 2$. Suppose (X, ℓ) is a general point in $I_k(m_1, \dots, m_k)$. Then the point-line correspondence $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ is mini-versal near the k -incident line $[\ell] \in \mathbf{G}(1, r)$.*

Proof. The map ρ in Lemma 1.24 is the differential of (any of) the induced maps

$$(U, [\ell]) \longrightarrow W_{m_1} \times \dots \times W_{m_k}$$

from Proposition 1.7 (Here U is an étale neighborhood of ℓ). The theorem follows immediately from Proposition 1.7. \square

Corollary 1.28. *Let $X \subset \mathbf{P}^r$ be a general degree $d \geq 2r - 1$ hypersurface, and let $k = d - (2r - 2)$. Then*

$$(\Sigma \subset \mathbf{P} \longrightarrow \mathbf{G}(1, r))$$

is versal at every k -incident line $[\ell] \in \mathbf{G}(1, r)$.

Proof. When $k = d - 2(r - 1)$, the irreducible variety $I_k(m_1, \dots, m_k)$ maps generically finitely onto \mathbf{P}^N . The locus consisting of pairs $(X, \ell) \in I_k(m_1, \dots, m_k)$ for which the point-line correspondence fails to be versal at ℓ is a strict (closed) subset of $I_k(m_1, \dots, m_k)$ by Theorem 1.27. Since this is true for all multiplicity data (m_1, \dots, m_k) , we conclude that there is a Zariski open subset of \mathbf{P}^N for which the corollary holds. \square

2 Maximal monodromy. The purpose of this section is to establish a basic result about the monodromy of lines meeting a general hypersurface at the least possible number of points.

Theorem 2.1. *Suppose $k = d - (2r - 2) \geq 2$, and suppose $\sum_{i=1}^k m_i = d$ is a partition with at least two distinct parts. Then the Galois group of the generically finite projection*

$$\varphi : I_k(m_1, \dots, m_k) \longrightarrow \mathbf{P}^N$$

is the full symmetric group.

Our approach to proving Theorem 2.1 is standard. Let G denote the Galois group of φ . We break the proof up into two standard steps, in increasing difficulty:

- (1) Show G is 2-transitive.
- (2) Show G contains a simple transposition.

These two properties imply that G is the full symmetric group. Step 1 is easy and quite standard, whereas Step 2 requires some care, and will occupy most of our attention.

2.1 2-transitivity.

Lemma 2.2. *The monodromy group G is 2-transitive.*

Proof. It suffices to show that the variety

$$I_k(m_1, \dots, m_k) \times_{\mathbf{P}^N} I_k(m_1, \dots, m_k) \setminus \Delta$$

is irreducible. (Here Δ is the diagonal.) By Lemma 1.13 we know that $I''_{k,k}(m_1, \dots, m_k | m_1, \dots, m_k)$ is irreducible. Since this correspondence dominates $I_k(m_1, \dots, m_k) \times_{\mathbf{P}^N} I_k(m_1, \dots, m_k) \setminus \Delta$, we conclude the claimed irreducibility, and hence the corollary. \square

Remark 2.3. We do not have to concern ourselves with incident pairs of lines ℓ, ℓ' – the reader can check that a general $X \in \mathbf{P}^N$ will not have an intersecting pair of k -incident lines by an elementary dimension count.

2.2 In pursuit of a simple transposition.

Notation and Setup 2.4. Until otherwise specified, we adopt the following notation:

- (1) $(X, \ell) \in I_{k-1}(m_1 + m_2, m_3, \dots, m_k)$ is a general point, with $k = d - (2r - 2) \geq 2$. Furthermore, we assume $m_1 \neq m_2$.
- (2) We let $\mathbf{A}^{m_1+m_2-1}$ denote the base of the mini-versal deformation $W_{m_1+m_2}$.
- (3) We let $Z_{m_1, m_2} \subset \mathbf{A}^{m_1+m_2-1}$ be the locus parametrizing divisors of the form $m_1 p + m_2 q$.
- (4) $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ denotes the point-line correspondence for X .

Lemma 2.5. Z_{m_1, m_2} is a uni-branched curve with a multiplicity two singularity at the origin $0 \in \mathbf{A}^{m_1+m_2-1}$.

Proof. The family of deformations of $z^{m_1+m_2} = 0$ parametrized by the set Z_{m_1, m_2} is the image of the map

$$t \mapsto (z - m_2 t)^{m_1} (z + m_1 t)^{m_2}.$$

By expanding this polynomial in terms of z , we obtain an expression

$$z^{m_1+m_2} + c_2 t^2 z^{m_1+m_2-2} + c_3 t^3 z^{m_1+m_2-3} + \dots + c_{m_1+m_2} t^{m_1+m_2}$$

where, under the assumption that $m_1 \neq m_2$, all coefficients c_j are nonzero. The lemma follows. \square

Corollary 2.6. Let $D \subset \mathbf{A}^{m_1+m_2-1}$ be a divisor given by the vanishing of an equation $f(a_2, \dots, a_{m_1+m_2}) = 0$, where f is such that $0 \in D$, and such that $\frac{\partial}{\partial a_2} f(0) \neq 0$. Further suppose D does not contain Z_{m_1, m_2} .

Then

$$\text{length}(D \cap Z_{m_1, m_2}) = 2.$$

Proof. This easily follows from the parametrization of Z_{m_1, m_2} found in the proof of Lemma 2.5. \square

Proposition 2.7. Let $(X, \ell) \in I_{k-1}(m_1 + m_2, m_3, \dots, m_k)$ be a general point, and let $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ be the point-line correspondence.

Denote by

$$v : (U, [\ell]) \longrightarrow (\mathbf{A}^{m_1+m_2-1} \times \mathbf{A}^{m_3-1} \times \dots \times \mathbf{A}^{m_k-1}, 0)$$

a local map to the mini-versal deformation space where $U \subset \mathbf{G}(1, r)$ is a sufficiently small étale neighborhood of $[\ell]$. Then

- (1) The map v is an immersion near $[\ell]$.
- (2) The image $v(U)$ is transverse to $Z_{m_1, m_2} \subset (\mathbf{A}^{m_1+m_2-1}, 0, \dots, 0)$ in the sense of Corollary 2.6.

Proof. The two statements are proved at the level of tangent spaces. They follow, respectively, by applying points 1 and 2 in Lemma 1.25. \square

2.2.1 “No unintended consequences” lemma. Now suppose

$$(X, \ell, \ell') \in I_{k-1, k}(m_1 + m_2, m_3, \dots, m_k \mid m_1, \dots, m_k)$$

is a general point. The next lemma and corollary show that the condition of possessing the $k - 1$ -incident line ℓ has no effect on the k -incident line ℓ' of X . Hence, there are in general “no unintended consequences elsewhere” for having a $k - 1$ -incident line.

Lemma 2.8. *The point-line correspondence*

$$\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$$

is mini-versal near $[\ell'] \in \mathbf{G}(1, r)$.

Proof. After using the PGL_{r+1} -action, we may assume

$$\ell = [x_0 : x_1]$$

$$\ell' = [x_2 : x_3]$$

and therefore, that the homogeneous ideal of the union $\ell \cup \ell'$ is

$$I_{\ell \cup \ell'} = (x_0 x_2, x_1 x_2, x_0 x_3, x_1 x_3, x_4, x_5, \dots, x_r).$$

Therefore, we may write the defining equation of X in the form

$$F = p(x_0, x_1) + q(x_2, x_3) + \sum_{i=0,1; j=2,3} x_i x_j A_{i,j} + \sum_{i=4}^r x_i G_i$$

where $A_{i,j} \in \Gamma(\mathcal{O}_{\mathbf{P}^r}(d-2))$, and $G_i \in \Gamma(\mathcal{O}_{\mathbf{P}^r}(d-1))$.

We let $a_{i,j} = A_{i,j}|_{\ell'}$, and $g_i = G_i|_{\ell'}$.

As in the proof of Lemma 1.21, we apply the infinitesimal substitutions

$$x_0 \mapsto x_0 + \varepsilon \ell_0$$

$$x_1 \mapsto x_1 + \varepsilon \ell_1$$

$$x_2 \mapsto x_2$$

$$x_3 \mapsto x_3$$

$$x_4 \mapsto x_4 + \varepsilon \ell_4$$

$$\vdots$$

$$x_r \mapsto x_r + \varepsilon \ell_r$$

and then mod out by the ideal $I_{\ell'} = (x_0, x_1, x_4, \dots, x_r)$. (Here ℓ_i are linear forms in the variables x_2 and x_3 .)

The result is the infinitesimal family

$$f_\varepsilon = q(x_2, x_3) + \varepsilon \ell_0(x_2 a_{0,2} + x_3 a_{0,3}) + \varepsilon \ell_1(x_2 a_{1,2} + x_3 a_{1,3}) + \sum_{i=4}^r \varepsilon \ell_i g_i.$$

Set $g_0 := x_2 a_{0,2} + x_3 a_{0,3}$ and $g_1 := x_2 a_{1,2} + x_3 a_{1,3}$. Then, general choices of the four polynomials $a_{i,j}$ yield general choices of g_0 and g_1 .

These observations tell us that the first-order analysis of $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ near $[\ell']$ is exactly the same as in the case for a general point $(X, \ell') \in I_k(m_1, \dots, m_k)$. In particular, since $g_0, g_1, g_4, \dots, g_r$ are taken to be generic, π is mini-versal near $[\ell']$. \square

Corollary 2.9. *Let $(X, \ell) \in I_{k-1}(m_1 + m_2, m_3, \dots, m_k)$ be a general point. Then the point-line correspondence*

$$\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$$

is mini-versal at all k -incident lines $[\ell'] \in \mathbf{G}(1, r)$.

Proof. Indeed, the variety $I''_{k-1,k}(m_1 + m_2, m_3, \dots, m_k \mid m_1, m_2, \dots, m_k)$ is irreducible, and generically finite over $I_{k-1}(m_1 + m_2, m_3, \dots, m_k)$. The previous lemma shows that there is a Zariski open subset of $I_{k-1,k}$ with ℓ' versal. The corollary follows. \square

Lemma 2.10. *Let $(X, \ell) \in I_{k-1}(n_1, \dots, n_{k-1})$ be a general point. Then X does not possess any other $k - 1$ -incident lines.*

Proof. This follows from a simple dimension count – we leave it to the reader. \square

Theorem 2.11. *The monodromy group G contains a simple transposition.*

Proof. Let $(X, \ell) \in I_{k-1}(m_1 + m_2, \dots, m_k)$ be a general point.

All k -incident lines of multiplicity (m_1, \dots, m_k) are simple, i.e. occur with multiplicity one, according to Corollary 2.9. The line $[\ell]$ occurs as a multiplicity 2 k -incident line of type (m_1, \dots, m_k) , according to Corollary 2.6.

Therefore, above the point $[X] \in \mathbf{P}^N$, the correspondence $I_k(m_1, \dots, m_k)$ has exactly one nonreduced point of length 2, and all other points are reduced.

Since the curve $Z_{m_1, m_2} \subset \mathbf{A}^{m_1 + m_2 - 1}$ is irreducible, the two nearby k -incident lines (for a nearby hypersurface X') limiting to $[\ell]$ experience monodromy.

Hence we obtain a simple transposition in the monodromy group G . \square

2.2.2 Moduli of lines meeting X infrequently.

Definition 2.12. Let $k \geq 4$. We define $i_{(m_1, \dots, m_k)} : I_k(m_1, \dots, m_k) \dashrightarrow \mathcal{M}_{0,k}/S_k$ to be the map sending (X, ℓ) to $[\ell \cap_{\text{set}} X \subset \ell]$. Here, subscript “set” means “set theoretic”.

Corollary 2.13. *Assume $k = d - (2r - 2) \geq 4$, and let X be a general degree d hypersurface. Then any two distinct $(X, \ell), (X, \ell') \in I_k(m_1, \dots, m_k)$ have distinct images under $i_{(m_1, \dots, m_k)}$.*

Proof. Let $J \subset I_k(m_1, \dots, m_k) \times_{\mathbf{P}^N} I_k(m_1, \dots, m_k)$ denote the closure of the locus of pairs $((X, \ell), (X, \ell'))$ having equal images under $i_{(m_1, \dots, m_k)}$, and such that $\ell \cap \ell'$ is empty.

By Lemma 2.2, The fiber product $I_k(m_1, \dots, m_k) \times_{\mathbf{P}^N} I_k(m_1, \dots, m_k) \setminus \Delta$ is irreducible, where Δ denotes the diagonal.

It is easy to exhibit two lines $\ell, \ell' \subset \mathbf{P}^r$ and a degree d hypersurface X such that ℓ and ℓ' are k -incident with multiplicity (m_1, \dots, m_k) and such that $i_{(m_1, \dots, m_k)}(\ell) \neq i_{(m_1, \dots, m_k)}(\ell')$.

Therefore, J , being a closed set, must have dimension strictly smaller than this fiber product’s dimension. But the latter is the dimension of \mathbf{P}^N , i.e. the locus J projects to a strict subset of \mathbf{P}^N . This proves the corollary. \square

3 Extending μ_1 across a k -incident line.

Notation and Setup 3.1. Throughout this section:

- (1) $X \subset \mathbf{P}^r$ is a general hypersurface of degree d .
- (2) We set $k := d - (2r - 2)$, and assume $k \geq 1$.
- (3) We let $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ be the point-line correspondence for X .
- (4) We continue to let

$$\mu_1 : \mathbf{G}(1, r) \dashrightarrow \text{Hyp}(d, 1)$$

denote the moduli map.

- (5) Finally, we let ℓ be a k -incident line of X with multiplicity (m_1, \dots, m_k) .

Remark 3.2. We will assume that the reader has some familiarity with the moduli space $\overline{\mathcal{M}}_{0,d}$. We will review some important background as appropriate.

Our goal is to understand the resolution of indeterminacy of μ_1 near ℓ .

The moduli space $\text{Hyp}(d, 1)$ is none other than the space $\mathcal{M}_{0,d}/S_d$. Unfortunately the latter moduli space is not fine, so we will first change settings to circumvent this technical inconvenience.

Definition 3.3. We define

$$\Sigma^{[d]} := \overline{\Sigma \times_{\pi} \cdots \times_{\pi} \Sigma \setminus \text{Diagonals}}$$

We let $s : \Sigma^{[d]} \longrightarrow \mathbf{G}(1, r)$ denote the structural degree $d!$ map.

Proposition 3.4. *The scheme $\Sigma^{[d]}$ is smooth at any point $[\hat{\ell}] \in s^{-1}([\ell])$.*

Proof. This is a consequence of Corollary 1.28. Indeed, étale locally around every $[\ell] \in \mathbf{G}(1, r)$, we may assume $\pi : \Sigma \longrightarrow \mathbf{G}(1, r)$ is isomorphic to products of maps of the form

$$\{(z, a_0, \dots, a_{n-2}) \mid z^n + a_{n-2}z^{n-2} + \dots + a_0 = 0\} \longrightarrow (a_0, \dots, a_{n-2})$$

If r_1, \dots, r_n denote the roots of $z^n + a_{n-2}z^{n-2} + \dots + a_0$, then the variety

$$\text{Spec } k[r_1, \dots, r_n] / (r_1 + \dots + r_n = 0)$$

maps to $\text{Spec } k[a_{n-2}, \dots, a_0]$ by the elementary symmetric expressions in the r_i .

This, in turn, means that the scheme $\Sigma^{[d]}$, above the point $[\ell] \in \mathbf{G}(1, r)$, is étale locally isomorphic to a product of affine spaces $\text{Spec } k[r_1, \dots, r_n] / (r_1 + \dots + r_n)$, and in particular smooth. \square

Lemma 3.5. *The set $s^{-1}([\ell])$ has $\frac{d!}{m_1!m_2!\dots m_k!}$ elements.*

Proof. The group S_d acts transitively on the set $s^{-1}([\ell])$. From the local description of s over $[\ell]$ explained in the proof of the previous proposition, we see that the stabilizer of a point in $s^{-1}([\ell])$ is a copy of $S_{m_1} \times \dots \times S_{m_k} \subset S_d$. \square

Definition 3.6. We let $\pi^{[d]} : \mathbf{P} \longrightarrow \Sigma^{[d]}$ denote the tautological \mathbf{P}^1 -bundle, pulled back from $\mathbf{G}(1, r)$. $\pi^{[d]}$ has d tautological sections σ_i , $i = 1, \dots, d$.

We let $U \subset \Sigma^{[d]}$ denote the open set over which none of the sections σ_i intersect one another. We obtain an induced moduli map which we write as:

$$\mu_1^{[d]} : U \longrightarrow \overline{\mathcal{M}}_{0,d}.$$

Finally, we let $[\hat{\ell}]$ be any point in $\Sigma^{[d]}$ lying over $[\ell]$.

3.1 Resolution near $[\hat{\ell}]$. Now that we have altered our setting, we begin the study of the resolution of indeterminacy of $\mu_1^{[d]}$, viewed as a rational map

$$\mu_1^{[d]} : \Sigma^{[d]} \dashrightarrow \overline{\mathcal{M}}_{0,d}$$

near the point $[\hat{\ell}]$.

Definition 3.7. We define

$$Z \subset \Sigma^{[d]} \times \overline{\mathcal{M}}_{0,d}$$

to be the closure of the graph of $m_1^{[d]}$. Furthermore, we let $\alpha : Z \rightarrow \overline{\mathcal{M}}_{0,d}$ denote the projection to the second factor and we let $\beta : Z \rightarrow \Sigma^{[d]}$ be the natural (birational) projection.

3.2 Blowup description of $\overline{\mathcal{M}}_{0,d}$. Recall that the moduli space $\overline{\mathcal{M}}_{0,d}$ can be obtained as a blow up

$$\kappa : \overline{\mathcal{M}}_{0,d} \rightarrow (\mathbf{P}^1)^{d-3}$$

which, to a general point $(\mathbf{P}^1, s_1, \dots, s_d) \in \overline{\mathcal{M}}_{0,d}$ assigns the coordinates of (s_4, \dots, s_d) once we use the PGL_2 -action to send s_1 to 0, s_2 to 1 and s_3 to ∞ .

Notation and Setup 3.8. Moving forward, we will use the following objects and notation:

- (1) Let $n \geq 1$ be any integer. We let Δ_n denote the scheme

$$\text{Spec } k[r_1, \dots, r_n] / (r_1 + \dots + r_n = 0).$$

- (2) We fix $\alpha_1, \dots, \alpha_k \in \mathbf{A}^1$, and corresponding multiplicities m_1, \dots, m_k such that $\sum_{i=1}^k m_i = d$.

- (3) We assume $k \geq 3$.

- (4) We define the *tautological family* F_n over Δ_n anchored at the point α_i to be the scheme

$$\prod_{j=1}^n (z - \alpha_i - r_j) = 0.$$

Note: F_n is a closed subscheme of $\Delta_n \times \mathbf{A}^1$.

- (5) Let $\varphi_n(\alpha_i) : \Delta_n \rightarrow (\mathbf{A}^1)^n$ be defined by the formula:

$$(r_1, \dots, r_n) \mapsto (\alpha_i + r_1, \dots, \alpha_i + r_n).$$

- (6) We let

$$\varphi_{m_1, \dots, m_k}(\alpha_1, \dots, \alpha_k) : \Delta_{m_1} \times \dots \times \Delta_{m_k} \rightarrow (\mathbf{A}^1)^d \subset (\mathbf{P}^1)^d$$

denote the map $\varphi_{m_1}(\alpha_1) \times \dots \times \varphi_{m_k}(\alpha_k)$.

- (7) Let $x \in (\mathbf{P}^1)^m$ be any point. We let D_x denote the vector space of first order deformations of x induced by the diagonal PGL_2 -action on $(\mathbf{P}^1)^m$.

Remark 3.9. Of course, $\Delta_n \simeq \mathbf{A}^{n-1}$, but we want to emphasize the particular presentation of the coordinate ring.

Proposition 3.10. *Under the assumptions above, the map*

$$\varphi_{m_1, \dots, m_k}(\alpha_1, \dots, \alpha_k) : \Delta_{m_1} \times \dots \times \Delta_{m_k} \rightarrow (\mathbf{P}^1)^d$$

has the following properties

- (1) *The induced map on tangent spaces is injective at $0 \in \Delta_{m_1} \times \dots \times \Delta_{m_k}$.*
(2) *The tangent space $T_0 \Delta_{m_1} \times \dots \times \Delta_{m_k}$ and the space $D_{\varphi_{m_1, \dots, m_k}(\alpha_1, \dots, \alpha_k)(0)}$ are linearly independent vector subspaces of the tangent space of $(\mathbf{P}^1)^d$ at $\varphi_{m_1, \dots, m_k}(\alpha_1, \dots, \alpha_k)(0)$.*

Proof. The first point is basically by definitions.

An element $v \in D_{\varphi_{m_1, \dots, m_k}(\alpha_1, \dots, \alpha_k)(0)}$ is determined by its effect at the three distinct points α_1, α_2 and α_3 in \mathbf{P}^1 . Assume the first order deformation v has the following effect:

$$\begin{aligned}\alpha_1 &\mapsto \alpha_1 + \varepsilon v_1 \\ \alpha_2 &\mapsto \alpha_2 + \varepsilon v_2 \\ \alpha_3 &\mapsto \alpha_3 + \varepsilon v_3.\end{aligned}$$

Due to the defining equation ($r_1 + \dots + r_n = 0$) for Δ_n , In order for v to be in $T_0 \Delta_{m_1} \times \dots \times \Delta_{m_k}$, we must have:

$$\begin{aligned}m_1 v_1 &= 0 \\ m_2 v_2 &= 0 \\ m_3 v_3 &= 0.\end{aligned}$$

But then all v_i are zero, and hence v is. This is what we wanted to show. \square

Corollary 3.11. *The induced map*

$$\bar{\varphi} : \Delta_{m_1} \times \dots \times \Delta_{m_k} \longrightarrow (\mathbf{P}^1)^d / \mathrm{PGL}_2 \simeq (\mathbf{P}^1)^{d-3}$$

is a local immersion near the point $0 \in \Delta_{m_1} \times \dots \times \Delta_{m_k}$.

Proof. This follows from Proposition 3.10. \square

Corollary 3.12. *There exists an étale neighborhood $V \subset \Sigma^{[d]}$ containing $[\hat{\ell}]$ such that*

$$\alpha : \beta^{-1}(V) \longrightarrow \overline{\mathcal{M}}_{0,d}$$

is an isomorphism onto its image. In particular, α is unramified at all points of $\beta^{-1}([\hat{\ell}])$.

Proof. This follows from Corollary 3.11, from the fact that $\Sigma^{[d]}$ is étale locally isomorphic to $\Delta_{m_1} \times \dots \times \Delta_{m_k}$ near $[\hat{\ell}]$, and the description of $\overline{\mathcal{M}}_{0,d}$ as a blowup $\kappa : \overline{\mathcal{M}}_{0,d} \longrightarrow (\mathbf{P}^1)^{d-3}$. \square

3.3 Stable reduction. For the reader's convenience, we briefly review the procedure of stable reduction for families of d -pointed rational curves.

Notation and Setup 3.13. The following notation will appear frequently in this section:

- (1) Let B be a smooth curve, and $b \in B$ a specified point.
- (2) Let $\sigma_1, \dots, \sigma_d$ be distinct sections of the projection $\mathbf{P}^1 \times B \longrightarrow B$. We assume that away from b , the sections σ_i are mutually disjoint. We let \mathbf{P}_b^1 denote $\mathbf{P}^1 \times \{b\}$.
- (3) Let

$$D := \left(\bigcup_i \sigma_i \right) \cap \mathbf{P}_b^1.$$

and write $D = p_1 + \dots + p_n + a_1 q_1 + \dots + a_{k-n} q_{k-n}$, with $a_i > 1$ and all points p_j, q_j distinct.

- (4) We denote stable genus 0 curves in the usual way as (P, s_1, \dots, s_d) . The nodes and marked points of (P, s_1, \dots, s_d) are called *special points*.

Proposition 3.14. *Assume $k \geq 3$. Then the stable replacement $(P, s_1, \dots, s_d) \in \overline{\mathcal{M}}_{0,d}$ of the pair $D \subset \mathbf{P}_b^1$ is a union of*

- (1) \mathbf{P}_b^1 and
- (2) $T_i, i = 1, \dots, k - n$, possibly-nodal rational tails with $T_i \cap \mathbf{P}_b^1 = q_i$.

The marked points $\{s_1, \dots, s_d\}$ are distributed as follows: on \mathbf{P}_b^1 the points p_1, \dots, p_n are marked, while each T_i contains a_i marked points.

Proof. We recall the stable reduction procedure, from which the proposition is immediately verified.

- (1) We view $C := \bigcup \sigma_i \subset \mathbf{P}^1 \times B$ as a curve on a surface, with singularities $q_i \in \mathbf{P}_b^1$.
- (2) We resolve the singularities $q_i \in C$ by blowing up repeatedly until all branches of C through q_i are separated.
- (3) Since each blow up in the resolution process happens at smooth centers, the new fiber P of the blown up surface over the point $b \in B$ is the union of the original \mathbf{P}_b^1 and several rational tails attached to \mathbf{P}_b^1 at the points q_i .
- (4) We contract any components of P which have ≤ 2 special points.

The proposition follows from this description of the stable reduction process. \square

Remark 3.15. The process of stable reduction described in the proof of Proposition 3.14 will be referred to frequently in the remainder of this section. We emphasize that when $k \geq 3$, the original fiber \mathbf{P}_b^1 naturally persists as a component of the stable reduction P .

Definition 3.16. A *chain* is a nodal curve $T = P_1 \cup \dots \cup P_j$ where each $P_i \equiv \mathbf{P}^1$, and P_i intersects P_{i+1} at one node, for all $i = 1, \dots, j-1$, with no further intersections between components.

Proposition 3.17. Assume $(P, s_1, \dots, s_d) \in \overline{\mathcal{M}}_{0,d}$ is a stable curve which is the nodal union of:

- (1) a \mathbf{P}^1 with n marked points p_1, \dots, p_n , and
- (2) $k - n$ chains T_i , $i = 1, \dots, k - n$, with each T_i intersecting \mathbf{P}^1 once at a point q_i , every component of T_i having exactly three special points, and each T_i containing $a_i > 1$ marked points.

Suppose (P, s_1, \dots, s_d) is the stable replacement for $D \subset \mathbf{P}_b^1$ in Notation and Setup 3.13.

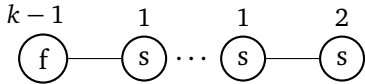
Then

- (1) $D \subset \mathbf{P}_b^1$ is supported on $\leq k$ points.
- (2) If D is supported on k points, and if we assume $k \geq 4$, then D must equal $p_1 + \dots + p_n + a_1 q_1 + \dots + a_{k-n} q_{k-n}$.

Proof. These two points follow from the stable reduction procedure described in the proof of Proposition 3.14, and from the observations in Remark 3.15. \square

3.4 Dandelions and the proof of Theorem 0.1.

Definition 3.18. Any point $[(P, s_1, \dots, s_d)] \in \overline{\mathcal{M}}_{0,d}$ having dual graph



is called a *k-dandelion*. There are a total of $d - k$ vertices labelled “s”. (Here, the numbers above a vertex indicate the number of marked points on the corresponding component of the stable curve.) The *stem* of the dandelion is the curve which is the union of components labelled “s”. The *flower* of the *k-dandelion* is the component corresponding to the vertex labelled “f”.

Proposition 3.19. Let $[P] \in \alpha(Z)$ be a *k-dandelion* with $k = d - (2r - 2) \geq 4$. Then the map α is unramified at every preimage of $[P]$. Furthermore, every preimage of $[P]$ projects, under β to a point in $\Sigma^{[d]}$ lying over a *k*-incident line $[\ell] \in \mathbf{G}(1, r)$ with multiplicity $(d - k + 1, 1, \dots, 1)$.

Proof. The hypersurface X does not have any k' -incident lines with $k' < k$. Therefore, the proposition follows from point 2 of Proposition 3.17. \square

Proposition 3.20. *Suppose $k = d - (2r - 2) \geq 3$. Then there exists a k -dandelion in the image of α .*

Proof. Consider the one-parameter family of polynomials $(z - t)(z - t^2) \dots (z - t^{d-k})(z + t + t^2 + \dots + t^{d-k})$ viewed as a map $\mathbf{A}^1 \rightarrow \Delta_{d-k+1}$ given by $(r_1, \dots, r_{d-k+1}) = (t, t^2, \dots, t^{d-k}, -t - t^2 - \dots - t^{d-k})$. If ℓ is a k -incident line of X with multiplicities $(d - k + 1, 1, \dots, 1)$ then Δ_{d-k+1} is a local chart for any point $[\hat{\ell}] \in \Sigma^{[d]}$ lying above ℓ , and therefore the one-parameter family above is realized, étale locally, as a curve on $\Sigma^{[d]}$. Applying stable reduction to this one parameter family yields a k -dandelion, as the reader can easily verify. \square

We have all ingredients to prove the $m = 1$ case of Theorem 0.1.

Theorem 3.21. *Suppose $d \geq 2r + 2$, and $X \subset \mathbf{P}^r$ is a general degree d hypersurface. Then the moduli map*

$$\mu_1 : \mathbf{G}(1, r) \dashrightarrow \text{Hyp}(d, 1)$$

is generically injective.

Proof. By Proposition 3.20, we know that there exists at least one k -dandelion $[P] \in \alpha(Z)$.

If $[P] \in \alpha(Z)$ is a k -dandelion, we have shown: The preimage $\alpha^{-1}([P])$ originates from a unique k -incident line ℓ , read off from the cross-ratios of the flower (Corollary 2.13), and the marking on the flower and stem of $[P]$ determines a point $[\hat{\ell}]$ and a point $z \in \beta^{-1}([\hat{\ell}])$, respectively, showing that $\alpha^{-1}([P]) = z$.

We conclude from Proposition 3.19 that the preimage of $[P]$ is the reduced point z , and from this the theorem follows. \square

We now give the proof of Theorem 0.1.

Proof of Theorem 0.1. By Theorem 3.21, we know that the map $\mu_1 : \mathbf{G}(1, r) \dashrightarrow \text{Hyp}(d, 1)$ is generically injective.

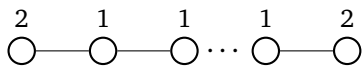
Let $\Lambda \subset \mathbf{P}^r$ be a general m -plane. The injectivity of μ_1 implies that a general line $\ell \subset \Lambda$ will be general in the sense that there will be no other line ℓ' with $[X \cap \ell \subset \ell] \simeq [X \cap \ell' \subset \ell']$.

Therefore, if Λ' is another m -plane with $[X \cap \Lambda \subset \Lambda] \simeq [X \cap \Lambda' \subset \Lambda']$, it must be that $\ell \subset \Lambda \cap \Lambda'$. Since this is true for a general $\ell \subset \Lambda$, we conclude that $\Lambda = \Lambda'$, which proves the theorem. \square

3.5 Straight trees and the proof of Theorem 0.3.

Notation and Setup 3.22. During this section, we assume $d = 2r + 1$ and $X \subset \mathbf{P}^r$ a general hypersurface of degree d .

Definition 3.23. A *straight tree* is any stable curve (P, s_1, \dots, s_d) having dual graph of the form



(There are a total of $d - 2$ vertices.)

Lemma 3.24. *Assume $d \geq 4$. There are $d!/8$ distinct straight trees.*

Proof. Indeed, we may exchange the two points on each extremal component of a straight tree, or we may flip the entire tree over, without altering the moduli of the straight tree. Therefore, we must divide $d!$ by $2 \times 2 \times 2$. \square

Lemma 3.25. *Suppose $z \in Z$ is such that $\alpha(z) \in \overline{\mathcal{M}}_{0,d}$ is a straight tree. Then $[\hat{\ell}] := \beta(z) \in \Sigma^{[d]}$ lies over a 3-incident line ℓ . Furthermore, the multiplicity of ℓ is $(1, a, b)$, where $a + b = d - 1$.*

Proof. X possesses no 2-incident lines. The lemma follows from Proposition 3.17, and from the observation in Remark 3.15. \square

Lemma 3.26. *Let a, b be positive integers such that $a + b = d - 1$. Then for every 3-incident line ℓ of multiplicity $(1, a, b)$, and every point $[\hat{\ell}] \in \Sigma^{[d]}$ lying over ℓ , there are precisely $a!b!/4$ points $z \in \beta^{-1}([\hat{\ell}])$ such that $\alpha(z)$ is a straight tree.*

Proof. Let p, q be the points of multiplicity a and b on ℓ . Then the variety $\beta^{-1}([\hat{\ell}])$ is the collection of stable curves obtained by attaching all rational tails with a marked points to the point p , and similarly with q . The lemma follows, after recalling that the two extremal points in a straight tree may be exchanged without changing the moduli of the tree – this explains the division by 4. \square

Lemma 3.27. *For every 3-incident line ℓ of multiplicity $(1, 1, d - 2)$, and every point $[\hat{\ell}] \in \Sigma^{[d]}$ lying over ℓ , there are precisely $(d - 2)!/2$ points $z \in \beta^{-1}([\hat{\ell}])$ such that $\alpha(z)$ is a straight tree.*

Proof. The proof is completely analogous to the proof of the previous lemma. The only difference is that we are only attaching rational tails to *one* point on ℓ , hence we must divide by two only once. \square

Theorem 3.28. *For any multiplicity (a, b, c) , let $n_{a,b,c}$ denote the number of 3-incident lines to X having the given multiplicity. Then the degree of the map $\alpha : Z \rightarrow \overline{\mathcal{M}}_{0,d}$ is*

$$\deg \alpha = 2 \sum_{a \geq b > 1} n_{(a,b,1)} + 4n_{(d-2,1,1)}.$$

Furthermore, $\deg \alpha = \deg \mu_1$.

Proof. The last statement is clear.

By symmetry considerations, every straight tree $[P] \in \overline{\mathcal{M}}_{0,d}$ has the same number of preimages in Z , all preimages arise from 3-incident lines, by Lemma 3.25, and all preimages are unramified points of α , by Corollary 3.11. Above a 3-incident line ℓ with multiplicity $(a, b, 1)$, where $a, b > 1$, there are $\frac{d!}{a!b!}$ points in $s^{-1}([\ell])$, by Lemma 3.5. Each contributes $a!b!/4$ straight trees by Lemma 3.26. Therefore, we obtain $\frac{d!}{a!b!} \frac{a!b!}{4} = \frac{d!}{4}$ total straight trees arising from 3-incident lines with multiplicities $(a, b, 1)$, where $a, b > 1$. Since there are $\frac{d!}{8}$ straight trees total (Lemma 3.24), we see that each such line contributes 2 to the degree of α . The same argument, using Lemma 3.27 explains the coefficient 4 for lines of type $(d - 2, 1, 1)$. \square

4 Further questions. There are many remaining questions worth understanding.

For instance, there is the question of extending Theorem 0.2 to the *equi-multiplicity* setting, i.e. where the multiplicity vector is (m, m, \dots, m) for some integer m . Recall the situation for plane curves: The monodromy groups of flexes of a plane cubic, and of bitangents of a plane quartic are not the respective full symmetric groups. Does this special phenomenon persist in higher dimensions? The first open case is to determine the monodromy group of 5-tangent lines of a general quintic surface in \mathbf{P}^3 .

Remark 4.1. There is one instance in the literature worth noting. In [D’S88], it is argued that the monodromy group of the set of 4-incident lines of type $(2, 2, 2, 2)$ to an octic surface $X \subset \mathbf{P}^3$ is the full symmetric group. Unfortunately, it appears (to the author) that there is a small gap in the proof – in the proof of Lemma 2.8, a particular variety I_8 is claimed to be locally irreducible, *because it is irreducible*. This is false as it stands.

However, the author believes the result is still true, and that D’Souza’s argument can be made to work.

We have settled the enumerative question (3) raised in the introduction in the case $m = 1$ of lines. However, there are many more instances where the preconditions of question (3) are met. We give the following interesting open example:

Example 4.2. Let $X_4 \subset \mathbf{P}^4$ be a general quartic threefold. What is the degree of the rational map

$$\mu_2 : \mathbf{G}(2, 4) \dashrightarrow \mathcal{M}_3?$$

Remark 4.3. One possible approach, similar to that pursued in this paper, would be to count the number of “complete quadrilaterals” on X_4 – these are plane curves which are the union of four general lines. The stable reduction procedure would also have to be analyzed.

Other enumerative puzzles emerge.

Example 4.4. Let $X_6 \subset \mathbf{P}^2$ be a general plane sextic. Then the moduli map $\mu_1 : \mathbf{P}^{2*} \dashrightarrow \mathcal{M}_{0,6}/S_6$ is generically injective. Since the target is three dimensional, we expect the double-point locus of μ_1 to be a curve $Y \subset \mathbf{P}^{2*}$. Is this the case?

If so, the assignment $X \mapsto Y$ is a contra-invariant of ternary sextics – what are the order and degree of this contra-invariant?

Example 4.5. Cadman and Laza [CL08] show that the degree of μ_1 for a plane Fermat quintic is 150. What is the degree in higher dimension, e.g. for the Fermat septic surface in \mathbf{P}^3 ? We suspect that the answer is $7^3 \times 24$, the size of the automorphism group of the Fermat septic.

Finally, and perhaps most importantly, there remains the question of improving Theorem 1.27. Indeed, it should be the case that the point-line correspondence is versal at every k -incident line with $k \geq d - (2r - 2)$ for a general hypersurface X . In fact, there should be an analogous versality result for higher dimensional plane sections of X – this is the subject of future work.

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